Irreversible thermodynamics of multistep transitions

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Conditions under which the evolution equations of a multivariate system can be cast in a variational form are identified. A kinetic potential generating both the deterministic part of the evolution and the probabilistic properties in a suitably defined set of variables is derived and compared to the thermodynamic potentials. The results are illustrated on the kinetics of phase transitions involving intermediate metastable phases and chemical reactions giving rise to two or more intermediate species.

I. INTRODUCTION

Many physical systems obey evolution equations of the form [1]

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}) + \mathbf{F}(t) \tag{1}$$

Here $\mathbf{x}=(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ is the state vector; $\mathbf{v}=(\mathbf{v}_1,\ldots,\mathbf{v}_n)$ accounts for the "deterministic" part of the evolution, responsible for the eventual tendency of $\mathbf{x}$ toward an invariant set of phase space referred to as attractor; and $\mathbf{F}=(\mathbf{F}_1,\ldots,\mathbf{F}_n)$ is a set of random (Langevin) forces accounting for the effect of microscopic level processes, manifested in the form of thermodynamic fluctuations around the deterministic path.

When the system operates in the linear range of irreversible processes and the deterministic part of the evolution is not subjected to externally imposed systematic nonequilibrium constraints the above evolution equations take, in the absence of inertial effects, the universal form [2]

$$\frac{d\mathbf{x}}{dt} = -L \cdot \frac{\partial U}{\partial \mathbf{x}} + \mathbf{F}(t), \tag{2}$$

where U is one of the equilibrium thermodynamic potentials such as the Helmholtz or Gibbs free energy, and L is a symmetric positive definite $n \times n$ matrix referred to as the Onsager matrix or the matrix of phenomenological coefficients. It is to be noted that the time derivative of $\mathbf{x}$ in the left-hand side of Eq. (2) can be viewed as a set of fluxes associated with the various irreversible processes present, $-\partial U/\partial \mathbf{x}$ being the set of corresponding thermodynamic forces.

The best known example of Eq. (1) pertains to the limit where the random forces are discarded (mean field description) and the evolution laws are linearized around a unique stable state of thermodynamic equilibrium. As beautifully established in De Groot and Mazur's classic monograph [3], there exists then a linear transformation of the variables $\mathbf{x}$ mapping the ($\mathbf{x}$-independent) Hessian matrix of U into the unit matrix, under which the evolution takes the form

$$\frac{d\mathbf{x}'}{dt} = -L' \cdot \mathbf{x}', \tag{3a}$$

where L' is a positive definite symmetric matrix. As a corollary, the evolution of the transformed variables $\mathbf{x}'$ can be written in a variational form,

$$\frac{d\mathbf{x}'}{dt} = -\frac{\partial \Phi}{\partial \mathbf{x}'}, \tag{3b}$$

where the potential Φ is given by

$$\Phi = \frac{1}{2} \sum_{ij} L'_{ij} x'_i x'_j. \tag{3c}$$

The question of whether systems out of equilibrium beyond the aforementioned limiting case can still give rise to variational properties has attracted considerable attention in the literature. A number of results have been reported, two familiar forms of which are Onsager's and Prigogine's extremal properties of the entropy production at the steady state in terms of the fluxes and the forces, respectively [3]. Still, the extent to which the evolution equations themselves (rather than the final state to which they drive the system) derive from a potential function in the sense of Eq. (3b) remains open. As a matter of fact, in the most general case of a nonlinear system subjected to arbitrary nonequilibrium constraints, it is believed that such a property fails, at least at the level of a mean field description [4].

In the present work, conditions are identified under which an affirmative answer to this question can be found. The crux is to focus on Eq. (2) where L is still a positive symmetric matrix, U is now allowed to keep its full nonlinear dependence on $\mathbf{x}$, and the random forces satisfy a fluctuation-dissipation type of relationship. There exists, then, a suitable set of variables $\{z_i\}$ related to $\{x_i\}$ by a linear transformation, under which the evolution equations take the form

$$\frac{dz_i}{dt} = -\frac{\partial \Phi}{\partial z_i} + R_i(t), \tag{4}$$

where $\Phi(\{z_i\})$ is a *kinetic potential* [not to be confused with the thermodynamic potential U in Eq. (2)] and the Langevin

forces $R_i(t)$ are related to the set of $\{F_i(t)\}$ in Eq. (1) by the aforementioned linear transformation. The general formulation is presented in Sec. II. It is applied successively, in Secs. III and IV, to phase-transition-, nucleation-, and chemical-reaction-related problems. The case where the fluctuation-dissipation property is not satisfied is also considered. The main conclusions are summarized in Sec. V.

II. VARIATIONAL FORM OF EVOLUTION LAWS: FORMULATION

Our starting point is Eq. (2) under the additional assumption that the random forces **F** define a multivariate white noise process,

$$\langle F_i(t)F_j(t')\rangle = 2D_{ij}\delta(t-t'), \tag{5}$$

it being understood that D_{ij} are state and time independent (additive noise).

As is well known, under these conditions the stochastic variable **x** undergoes a diffusion process whose probability density P satisfies the Fokker-Planck equation [5]

$$\frac{\partial P}{\partial t} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} L_{ij}\frac{\partial U}{\partial x_j}P + \sum_{j=1}^{n} D_{ij}\frac{\partial P}{\partial x_j}\right). \tag{6}$$

We next place ourselves under the conditions of validity of the fluctuation-dissipation theorem, entailing that the diffusion matrix D and the Onsager matrix L are proportional. Equation (6) takes then the simplified form

$$\frac{\partial P}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}L_{ij}\left(\frac{\partial U}{\partial x_j}P + \epsilon\frac{\partial P}{\partial x_j}\right), \tag{7}$$

where ϵ is a small parameter related, depending on the case, to the thermal energy kT or to the inverse of the system size. As a by-product, one obtains from this relation, as expected, that the invariant probability density is of the form

$$P_s \approx \exp\left(-\frac{U}{\epsilon}\right), \tag{8}$$

in agreement with equilibrium statistical mechanics.

To extract information concerning the time-dependent properties from Eq. (7) we introduce a linear transformation of variables,

$$\mathbf{x} = A \cdot \mathbf{z}, \tag{9a}$$

$$\frac{\partial}{\partial \mathbf{x}} = \tilde{A}^{-1} \cdot \frac{\partial}{\partial \mathbf{z}}, \tag{9b}$$

$\tilde{A}$ being the transpose of matrix A, and set

$$U(\mathbf{x}(\mathbf{z})) = \Phi(\mathbf{z}), \tag{9c}$$

$$P(\mathbf{x}(\mathbf{z})) = \rho(\mathbf{z}). \tag{9d}$$

Substituting into the equation we obtain, after some straightforward manipulations,

$$\frac{\partial \rho}{\partial t} = \left(\tilde{A}^{-1} \cdot \frac{\partial}{\partial \mathbf{z}}\right) \cdot \left\{L \cdot \left[\left(\tilde{A}^{-1}\frac{\partial \Phi}{\partial \mathbf{z}}\rho + \epsilon\tilde{A}^{-1}\frac{\partial \rho}{\partial \mathbf{z}}\right)\right]\right\}$$

or, in more explicit form,

$$\frac{\partial \rho}{\partial t} = \sum_{jm} (A^{-1}L\tilde{A}^{-1})_{jm}\left(\frac{\partial}{\partial z_j}\frac{\partial \Phi}{\partial z_m}\rho + \epsilon\frac{\partial^2 \rho}{\partial z_j\,\partial z_m}\right). \tag{10}$$

Equation (10) features the matrix

$$\Lambda = A^{-1}L\tilde{A}^{-1}, \tag{11}$$

linked to L by a *congruent transformation*. As is well known, under such a transformation a symmetric matrix can be diagonalized and can actually, upon a further linear scaling, be reduced to the unit matrix. This requirement determines fully the transformation matrix A and, upon substituting into Eq. (10), one obtains

$$\frac{\partial \rho}{\partial t} = \sum_{j=1}^{n} \frac{\partial}{\partial z_j}\left(\frac{\partial \Phi}{\partial z_j}\rho + \epsilon\frac{\partial \rho}{\partial z_j}\right). \tag{12}$$

This is the standard form of the Fokker-Planck equation of a system whose deterministic part of the evolution is driven by a kinetic potential Φ and which is subjected to additive fluctuations of identical variance. A system of this kind undergoes a process of *isotropic* diffusion in phase space described by the Langevin equation [see also Eq. (4)]

$$\frac{dz_i}{dt} = -\frac{\partial \Phi}{\partial z_i} + R_i(t), \tag{13a}$$

where the random forces satisfy the properties

$$\langle R_i(t)\rangle = 0, \quad \langle R_i(t)R_j(t')\rangle = \epsilon\delta_{ij}^{kr}\delta(t-t'). \tag{13b}$$

As a corollary the mean values $\langle z_i\rangle$ evolve according to

$$\frac{d\langle z_i\rangle}{dt} = -\left\langle \frac{\partial \Phi}{\partial z_i}\right\rangle \tag{13c}$$

and the invariant probability density is given by

$$\rho_s(\mathbf{z}) \approx \exp\left(-\frac{\Phi(\mathbf{z})}{\epsilon}\right). \tag{13d}$$

Equations (13) constitute the main result of this section, namely, the possibility of casting the evolution equations of a multivariate system in a form deriving from a kinetic potential under the assumption of detailed balance. They generalize the De Groot–Mazur formulation [Eqs. (3)] in two respects. First, nonlinear terms in the deviation from the reference state are retained. Such terms are indispensable in, among others, nucleation- and phase-transition-related problems, where the thermodynamic potential U possesses more than one minimum. Second, the presence of fluctuations is accounted for. Fluctuation-driven effects are crucial when transitions between states are taking place, as is the case in nucleation or more generally in systems giving rise to multiple states. It should be pointed out that the conjunction of fluctuations and nonlinearities implies that [contrary to Eqs. (3a)] Eqs. (13c) are not closed with respect to $\langle z_j\rangle$, but involve higher moments of the probability density as well. It is

only when the mean field approximation can be justified that these latter equations reduce to a closed set of deterministic evolution laws. Otherwise one needs to resort to the full Eqs. (13a).

We close this section by some comments on the case where the fluctuation-dissipation theorem does not hold. There are two broadly different mechanisms at the origin of such a failure: the evolution vector $\mathbf{v}$ in Eq. (1) contains an additional contribution beyond the one involving the thermodynamic potential; and/or the diffusion matrix D_{ij} reflects the fact that a subsystem that would otherwise be driven by the first term in Eq. (2) is embedded in a nonequilibrium medium. Focusing for concreteness on this second case, we have to account for the fact that the matrices L and D in Eq. (6) are distinct and cannot be reduced simultaneously to the unit matrix by a linear transformation. Nevertheless, one may still apply the transformation in Eqs. (9) to the matrix L alone, in which case Eq. (6) becomes

$$\frac{\partial \rho}{\partial t} = \sum_j \frac{\partial}{\partial z_j} \left(\frac{\partial \Phi}{\partial z_j} \rho + \epsilon \sum_m D'_{jm} \frac{\partial \rho}{\partial z_m} \right) \quad (14a)$$

with

$$\epsilon D' = A^{-1} D \tilde{A}^{-1}. \quad (14b)$$

Multiplying both sides of Eq. (14a) by z_i and integrating over all z variables one obtains a relation similar to (13c),

$$\frac{d\langle z_i \rangle}{dt} = -\left\langle \frac{\partial \Phi}{\partial z_i} \right\rangle. \quad (15)$$

In contrast, Eq. (14a) does not now admit a solution of the form (13d). On the other hand, in the limit of small ϵ the probability density can be expressed as [1,6]

$$\rho(t) \approx \exp\left(-\frac{\Psi(t)}{\epsilon} \right), \quad (16)$$

where to the dominant order in ϵ the stochastic potential Ψ satisfies a Hamilton-Jacobi type equation,

$$\frac{\partial \Psi}{\partial t} = \sum_i \left(\frac{\partial \Phi}{\partial z_i} \frac{\partial \Psi}{\partial z_i} - \epsilon \sum_j D'_{ij} \frac{\partial \Psi}{\partial z_i} \frac{\partial \Phi}{\partial z_j} \right). \quad (17)$$

We thus arrive at a representation where the evolution at the mean field level is governed by the kinetic potential Φ [Eq. (15)] whereas the fluctuations in general and the transitions between states in particular are governed by a quite different function, the stochastic potential Ψ. Both potentials are in general different from the equilibrium thermodynamic potentials U, although Φ is the transform of U under the change of variables in (9a).

III. MULTISTEP NUCLEATION

One instance in which the formulation outlined in the preceding section finds an interesting application pertains to first-order transitions mediated by intermediate metastable phases. Transitions of this kind are known to occur in a variety of materials, from protein solutions to aerosols to plasma crystals. Of special relevance for our purposes is protein crystallization, where the weakness and short-range character of the attractive part of the interactions favors the existence of a long-living metastable phase in the form of a high-concentration liquid. This phase tends to enhance significantly, under certain conditions, the rate of nucleation of crystals [7–9]. A number of studies of the above nonstandard, *two-step* nucleation problem, devoted to the construction of the free energy landscape and the computation of the associated activation barriers, have been reported [10]. Here we focus on the kinetic aspects of the process and, in particular, on the role of the kinetic potential Φ and its relationship with the free energy. For this purpose nucleation is formulated as the dynamics of a set of two order parameters, density x_1 and crystallinity (structure) x_2, evolving in an effective force field provided by the first part of Eq. (2) and subjected to thermal fluctuations in the form of Gaussian white noise [9],

$$\frac{dx_1}{dt} = -L_1 \frac{\partial U}{\partial x_1} - L \frac{\partial U}{\partial x_2} + F_1(t),$$

$$\frac{dx_2}{dt} = -L \frac{\partial U}{\partial x_1} - L_2 \frac{\partial U}{\partial x_2} + F_2(t). \quad (18)$$

We stress that in the absence of the intermediate phase Eqs. (18) collapse into a single equation for a unique order parameter, which can be cast straightforwardly in a variational form.

Switching to the Fokker-Planck description and assuming that the fluctuation-dissipation theorem remains valid [Eq. (7) with $n=2$] we seek for a congruent transformation (11) diagonalizing L with a matrix A of the form $A_{ii}=a_i$, $A_{ij}=A_{ji}=a$. Upon carrying out the algebra one obtains

$$a_1 = \frac{L_1 \pm \sqrt{L_1 L_2 - L^2}}{L} a, \quad a_2 = \frac{L_2 \pm \sqrt{L_1 L_2 - L^2}}{L} a \quad (19a)$$

and the Fokker-Planck equation in the z_1, z_2 variables,

$$\frac{\partial \rho}{\partial t} = \frac{\sqrt{L_1 L_2 - L^2} L^2}{a^2 (L_1 + L_2 + 2\sqrt{L_1 L_2 - L^2})}$$
$$\times \left[\frac{\partial}{\partial z_1} \frac{\partial \Phi}{\partial z_1} \rho + \frac{\partial}{\partial z_2} \frac{\partial \Phi}{\partial z_2} \rho + \epsilon \left(\frac{\partial^2 \rho}{\partial z_1^2} + \frac{\partial^2 \rho}{\partial z_2^2} \right) \right]. \quad (19b)$$

Notice that the argument in the square root is positive, owing to the positive definiteness of the matrix L. This relation can be further reduced to the form of Eq. (12) by rescaling the variables z_1, z_2 or, more straightforwardly, the time,

$$t = \tau \frac{a^2 (L_1 + L_2 + 2\sqrt{L_1 L_2 - L^2})}{L^2 \sqrt{L_1 L_2 - L^2}}, \quad (19c)$$

yielding

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial z_1}\frac{\partial \Phi}{\partial z_1}\rho + \frac{\partial}{\partial z_2}\frac{\partial \Phi}{\partial z_2}\rho + \epsilon\left(\frac{\partial^2 \rho}{\partial z_1^2} + \frac{\partial^2 \rho}{\partial z_2^2}\right). \quad (20)$$

In the context of the two-step nucleation considered in the present section, the free energy U possesses three minima located on the dilute and dense fluid phases I and II, and on the solid phase III. Now, from Eq. (9c), one has

$$\frac{\partial U}{\partial x_i} = \sum_{k=1}^{2} \widetilde{A}_{ik}^{-1}\frac{\partial \Phi}{\partial z_k},$$

$$\frac{\partial^2 U}{\partial x_i \, \partial x_j} = \sum_{k,\ell=1}^{2} \widetilde{A}_{ik}^{-1} A_{\ell j}^{-1}\frac{\partial^2 \Phi}{\partial z_k \, \partial z_\ell}. \quad (21)$$

The first of these relations implies that the extrema of U transform into extrema of Φ, since the matrix A is nonsingular. The second relation implies that the Hessians of U and Φ (in their respective variables) are related by the congruent transformation used to reduce L to the unit matrix which, by construction, preserves positivity. The nature of the extrema of U is thus not affected in the evolution in terms of the transformed variables. In short, at the level of Eqs. (19) and (20), the nucleation of crystals of the solid phase is reflected by fluctuation-induced transitions removing the system, initially located in a minimum associated with the fluid phase, toward a minimum associated with the solid phase. In a system whose dynamics is driven by a potential Φ as in Eq. (20) and in the limit of small ϵ (as is the case in situations of interest), the characteristic rate of these transitions is given by an extension of the classical Kramers theory [9,11,12], leading to (upon switching back to the original time units)

$$k = \frac{1}{2\pi}\left(\frac{\sigma_u^+}{|\sigma_u^-|}\right)^{1/2}(\sigma_{s1}\sigma_{s2})^{1/2}\frac{L^2\sqrt{L_1 L_2 - L^2}}{a^2(L_1 + L_2 + 2\sqrt{L_1 L_2 - L^2})}$$

$$\times \exp\left(-\frac{\Delta\Phi}{\epsilon}\right). \quad (22)$$

Here σ_{s1} and σ_{s2} are the eigenvalues of the Hessian matrix of Φ at the stable state of reference. As for $\sigma_u^\pm$ they are, respectively, the unstable and stable eigenvalues of this matrix evaluated at the transition state connecting the above state to one of the other stable states available. This transition state is known to correspond to a saddle point of the dynamical system $(z_1 \, z_2)$ in the mean field limit. Finally $\Delta\Phi$ is the potential barrier—the difference between the values of Φ at the transition and at the stable reference states. Now, by virtue of Eq. (9c) the potential barriers in the description in terms of U and Φ are bound to be equal. In contrast, the preexponential factors in Eq. (22) contain kinetic coefficients beyond the Hessian of U. We thus arrive at the conclusion that the barriers for nucleation are those given by the free energy, but the overall nucleation rate depends (weakly) on the properties of the kinetic potential Φ through the preexponential factors in Eq. (22). The situation is of course entirely different when the conditions of validity of the fluctuation-dissipation relationship are not met, in which case nucleation barriers would be governed by the stochastic potential Ψ [Eq. (17)] and thus take quite different values from the free energy barriers.
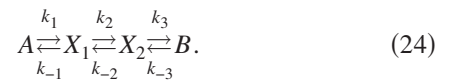
An interesting question from the standpoint of irreversible thermodynamics is whether the processes associated with the density and crystallinity (structure) fields can be thermodynamically coupled, as implied in Eq. (18) by the presence of the off-diagonal element L of the Onsager matrix. Now density is a true scalar whereas structure is accounted for by a tensor. At first sight, in an isotropic medium, this would rule out a thermodynamic coupling on the grounds of the Curie symmetry principle [3]. On the other hand a tensor T can be split in the following way:

$$T = \frac{1}{3}I \, \mathrm{tr}\, T + T^{(a)} + T^{(s)}, \quad (23)$$

where I is the unit tensor, $\mathrm{tr}\, T$ the trace of T which is a scalar, $T^{(a)}$ its antisymmetric part which can alternatively be viewed as an axial ("pseudo") vector, and $T^{(s)}$ its symmetric traceless part which can alternatively be viewed as a polar ("true") vector. Clearly, then, the density field x_1 can legitimately be coupled to the first part of the decomposition in Eq. (23) or, alternatively, to any other structure-related quantity x_2 of scalar nature such as the first coefficient of a Fourier series expansion of the solid phase density. The terms in L at the level of Eq. (18) stand, precisely, for this type of coupling. Notice that even in the absence of a nondiagonal term L in this equation, x_1 and x_2 would still be coupled kinetically, through the x_1 and x_2 dependence of the potential.

IV. CHEMICAL REACTIONS INVOLVING TWO OR MORE INTERMEDIATES

Chemical reactions constitute another important class of phenomena where the questions of coupling between variables, detailed balance, and nonequilibrium constraints are prominent. In this section we apply the procedure outlined in Sec. II in a reaction scheme describing the conversion of an initial reactant A to a final product B via the intermediates $X_1, \ldots, X_n$. Since all interesting aspects of the problem are already apparent when $n=2$, we shall in fact limit ourselves to

$$A \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} X_1 \underset{k_{-2}}{\overset{k_2}{\rightleftarrows}} X_2 \underset{k_{-3}}{\overset{k_3}{\rightleftarrows}} B. \quad (24)$$

This scheme can also be viewed as a "chemical analog" of the two-step nucleation considered in Sec. III, A and B being the analogs of the dilute fluid and solid phases, respectively. It will be assumed that there is no clearcut time scale separation in the evolution equations of X_1 and X_2 such as occurs for instance in the vicinity of a bifurcation point, allowing for the reduction of the dynamics to a single "normal form" equation, which can always be written in a variational form [13]. As an additional simplification we shall consider only first-order kinetics, since our aim is primarily to bring out the thermodynamic trends underlying the evolution.

A. Mean field analysis

The rate equations corresponding to scheme (24) admit a single steady state solution (we use the same symbol to denote a reactant and its concentration)

$$X_{1s} = \frac{1}{k_{-1}k_{-2} + k_{-1}k_3 + k_2k_3}[k_1(k_{-2} + k_3)A + k_{-2}k_{-3}B],$$

$$(25a)$$

$$X_{2s} = \frac{1}{k_{-1}k_{-2} + k_{-1}k_3 + k_2k_3}[k_1k_2A + k_3(k_{-1} + k_2)B].$$

$$(25b)$$

In the limit where the system operates around the state of thermodynamic equilibrium, the property of detailed balance (which in the presence of fluctuations would also guarantee the validity of the fluctuation-dissipation relationship; see below) implies that in each of the three independent reactions (24) the rates of the direct and inverse steps counteract each other. This requires that the concentrations A and B be linked by

$$\left(\frac{B}{A}\right)_{eq} = \frac{k_1k_2k_3}{k_{-1}k_{-2}k_{-3}}, \qquad (26a)$$

in which case Eqs. (25) collapse to

$$X_{1s} \equiv X_{1eq} = \frac{k_1A}{k_{-1}}, \quad X_{2s} \equiv X_{2eq} = \frac{k_{-3}B}{k_3}, \qquad (26b)$$

with

$$k_2X_{1eq} = k_{-2}X_{2eq}. \qquad (26c)$$

We turn next to the time evolution. Introducing the excess variables $x_1 = X_1 - X_{1s}$, $x_2 = X_2 - X_{2s}$, one obtains in the mean field limit,

$$\frac{dx_1}{dt} = -(k_{-1} + k_2)x_1 + k_{-2}x_2 = v_1,$$

$$\frac{dx_2}{dt} = k_2x_1 - (k_{-2} + k_3)x_2 = v_2. \qquad (27)$$

To cast these equations in the form of the mean field limit of Eq. (2), one needs to identify the appropriate fluxes and their conjugate thermodynamic forces. This has to be done by appealing to the bilinear form of the entropy production, which for scheme (24) becomes [3,4]

$$\sigma = w_1(\mu_A - \mu_{X_1}) + w_2(\mu_{X_1} - \mu_{X_2}) + w_3(\mu_{X_2} - \mu_B),$$

μ being the chemical potential and w_i the rates of the three reactions, $w_1 = k_1A - k_{-1}X_1$, $w_2 = k_2X_1 - k_{-2}X_2$, and $w_3 = k_3X_2 - k_{-3}B$. Adding and subtracting the values of the chemical potentials at the reference state (25) [or (26) in the equilibrium limit] and switching to the excess variables x_1 and x_2, one obtains

$$\sigma = w_1(\mu_A - \mu_{X_{1s}}) + (\mu_{X_{1s}} - \mu_{X_1})(w_1 - w_2)$$

$$+ (\mu_{X_{2s}} - \mu_{X_2})(w_2 - w_3) + w_3(\mu_{X_{2s}} - \mu_B). \qquad (28)$$

Now $w_1 - w_2$ and $w_2 - w_3$ are (in the excess variables) nothing but the rates $dx_1/dt = v_1$ and $dx_2/dt = v_2$ [Eqs. (27)]. When viewed as thermodynamic fluxes these quantities admit then, according to Eq. (28), $\mu_{X_{1s}} - \mu_{X_1}$ and $\mu_{X_{2s}} - \mu_{X_2}$ as conjugate

thermodynamic forces. In the limit of an ideal system, these differences reduce to the dominant order in x_1 and x_2 to $-x_1/X_{1s}$ and $-x_2/X_{2s}$, respectively. This suggests rewriting Eqs. (27) as

$$\frac{dx_1}{dt} = (k_{-1} + k_2)X_{1s}\frac{-x_1}{X_{1s}} - k_{-2}X_{2s}\frac{-x_2}{X_{2s}} = v_1,$$

$$\frac{dx_2}{dt} = -k_2X_{1s}\frac{-x_1}{X_{1s}} + (k_{-2} + k_3)X_{2s}\frac{-x_2}{X_{2s}} = v_2, \qquad (29)$$

where X_{1s} and X_{2s} are given by Eqs. (25).

These relations have the same structure as the deterministic part of Eqs. (2),

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -L \cdot \frac{\partial U}{\partial \mathbf{x}}, \qquad (30a)$$

where the thermodynamic potential U and the matrix L are, respectively,

$$U = \frac{x_1^2}{2X_{1s}} + \frac{x_2^2}{2X_{2s}}, \qquad (30b)$$

$$L = \begin{pmatrix} (k_{-1} + k_2)X_{1s} & -k_{-2}X_{2s} \\ -k_2X_{1s} & (k_{-2} + k_3)X_{2s} \end{pmatrix}. \qquad (30c)$$

If in addition detailed balance holds [Eq. (26c)], the matrix L becomes symmetric, as should be expected from the Onsager reciprocity relations. In this limit the potential U is still given by Eq. (30a), except that X_{1s} and X_{2s} stand for the equilibrium concentrations [Eqs. (26b)]. In other words, at the level of U, nonequilibrium constraints are manifested entirely through the values of the steady state concentrations, the structure of U itself being left the same as in equilibrium. This "local equilibrium" property is a consequence of the linearity of scheme (24).

B. Fokker-Planck equation and fluctuation-dissipation relation

We turn next to the properties of the fluctuations, generated by the second term in the right-hand side of Eq. (2). Since chemical reactions belong to the class of jump processes, the fundamental probabilistic description of a system like that in Eq. (24) is afforded by the master equation,

$$\frac{dP(X_1, X_2)}{dt} = \sum_{X_1', X_2'} \sum_{r=1}^{3} [W_r(X_1', X_2' \rightarrow X_1, X_2)P(X_1', X_2', t)$$

$$- W_r(X_1, X_2 \rightarrow X_1', X_2')P(X_1, X_2, t)], \qquad (31)$$

where W_r represents the probability per unit time of a transition between states X_1, X_2, and the states X_1' and X_2' connected to it by reaction step r. Expanding Eq. (31) around the reference state [Eqs. (25)] one obtains, in the limit of a large system, a Fokker-Planck equation [Eq. (6)] for the probability density $P(x_1, x_2, t)$ of the continuous variables x_1 and x_2 in which the drift part is given by Eqs. (30) and the diffusion matrix D_{ij} is given by the matrix of the second moments of the transition probabilities W_r [1,5]. One obtains after a straightforward calculation

$$D_{11} = \frac{\epsilon}{2}[k_1 A + (k_{-1} + k_2)X_{1s} + k_{-2}X_{2s}],$$

$$D_{22} = \frac{\epsilon}{2}[k_2 X_{1s} + (k_{-2} + k_3)X_{2s} + k_{-3}B],$$

$$D_{12} = D_{21} = -\frac{\epsilon}{2}(k_2 X_{1s} + k_{-2}X_{2s}), \tag{32}$$

where ϵ is the inverse of the system size. Comparing Eqs. (32) and (30c), one sees that the matrices L and D are distinct when X_{1s} and X_{2s} are evaluated in the presence of non-equilibrium constraints [Eqs. (25)] but become identical (up to an ϵ factor) in the limit where detailed balance holds [Eqs. (26)]. We here have a concrete illustration of the repercussions of the fluctuation-dissipation relation, as discussed in Sec. II.

C. Thermodynamic, kinetic, and stochastic potentials

Let us consider in some detail the case where detailed balance holds. Utilizing Eqs. (26) one may reduce the elements of the diffusion matrix [Eq. (32)] to

$$D_{11} = \epsilon(k_{-1} + k_2)X_{1eq},$$

$$D_{22} = \epsilon(k_{-2} + k_3)X_{2eq},$$

$$D_{12} = D_{21} = -\epsilon k_{-2}X_{2eq} = -\epsilon k_2 X_{1eq}. \tag{33}$$

The Fokker-Planck equation, which in the presence of detailed balance has a structure given by Eq. (7), takes the form [see also Eq. (29)]

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x_1}\left[\left(-(k_{-1}+k_2)X_{1eq}\frac{-x_1}{X_{1eq}} + k_{-2}X_{2eq}\frac{-x_2}{X_{2eq}}\right)P \right.$$
$$\left. + \epsilon(k_{-1}+k_2)X_{1eq}\frac{\partial P}{\partial x_1} - \epsilon k_{-2}X_{2eq}\frac{\partial P}{\partial x_2}\right]$$
$$+ \frac{\partial}{\partial x_2}\left[\left(k_2 X_{1eq}\frac{-x_1}{X_{1eq}} - (k_{-2}+k_3)X_{2eq}\frac{-x_2}{X_{2eq}}\right)P \right.$$
$$\left. + \epsilon(k_{-2}+k_3)X_{2eq}\frac{\partial P}{\partial x_2} - \epsilon k_2 X_{1eq}\frac{\partial P}{\partial x_1}\right], \tag{34}$$

displaying explicitly the proportionality (with the single factor ϵ) between the elements of the matrices L and D. It is now a simple matter to check that Eq. (34) admits a solution of the form (7) where U is given by Eq. (30b) evaluated at the equilibrium values of X_{1s} and X_{2s},

$$P(x_1, x_2) = (2\pi\epsilon X_{1eq}X_{2eq})^{-1/2} \exp\left[-\frac{1}{\epsilon}\left(\frac{x_1^2}{2X_{1eq}} + \frac{x_2^2}{2X_{2eq}}\right)\right]. \tag{35}$$

Now the exponent in Eq. (35) is nothing but the excess free energy of an ideal two-component system around its equilibrium value, which is in turn the thermodynamic potential of a system at constant temperature and volume. We thus see

that the potential involved in the deterministic part of the evolution in the sense of Eq. (30a), and the one generating the statistical properties of the system—referred to in Sec. II as the stochastic potential—are identical. Still, the deterministic equations are not driven by the potential in question, since the velocity vector is not the gradient of a scalar potential. To achieve this further reduction, we resort again to a congruent transformation diagonalizing L [Eq. (30c) with X_{is} replaced by $X_{i\,eq}$]. Choosing a matrix A of the form $A_{ii}=a_i$, $A_{ij}=A_{ji}=a$ and proceeding as in Sec. III, we obtain

$$\frac{a_1}{a} = 1 + r_1 + \sqrt{r_1 + r_2 + r_1 r_2}, \quad \frac{a_2}{a} = 1 + r_2 + \sqrt{r_1 + r_2 + r_1 r_2}, \tag{36a}$$

where r_1 and r_2 are the ratios of the rate constants for the two pathways of depletion of X_1 and X_2, respectively,

$$r_1 = \frac{k_{-1}}{k_2}, \quad r_2 = \frac{k_3}{k_{-2}}. \tag{36b}$$

Switching from the (x_1, x_2) to (z_1, z_2) variables according to Eqs. (9), we arrive then at a description in which, similarly to Eq. (20), the full dynamics is generated entirely by a kinetic potential Φ linked to the thermodynamic potential U by Eq. (9c),

$$\frac{\Phi}{a^2} = \left[\frac{1}{2X_{1eq}}\left(\frac{a_1}{a}\right)^2 + \frac{1}{2X_{2eq}}\right]z_1^2 + \left[\frac{1}{2X_{1eq}} + \frac{1}{2X_{2eq}}\left(\frac{a_2}{a}\right)^2\right]z_2^2$$
$$+ \left(\frac{1}{X_{1eq}}\frac{a_1}{a} + \frac{1}{X_{2eq}}\frac{a_2}{a}\right)z_1 z_2, \tag{37}$$

where a merely plays the role of a scaling factor. Contrary to the thermodynamic potential, this quantity displays a (kinetic) coupling between the two new variables, as well as coefficients depending on the rate constants. Theses coefficients cannot be reduced to equilibrium quantities like equilibrium constants, since the ratios r_1 and r_2 involve rate constants associated with two different reactions.

We finally turn to the case where the system is subjected to permanent nonequilibrium constraints compromising the validity of detailed balance. The Fokker-Planck equation has now the general structure of Eq. (6) in which the matrices L and D are given by Eqs. (30c) and (32) with X_{is} evaluated at the nonequilibrium steady state [Eq. (25)]. Nevertheless, as can be checked straightforwardly, it still admits a solution of the form of Eq. (35) except that $X_{i\,eq}$ are replaced by their nonequilibrium counterparts X_{is}. The stochastic potential (in the terminology of Sec. II) featured in this solution,

$$\Psi = \frac{x_1^2}{2X_{1s}} + \frac{x_2^2}{2X_{2s}}, \tag{38}$$

is identical to the potential U in Eq. (30b) featured in the deterministic part of the evolution, thereby illustrating once again the local equilibrium property alluded to at the end of Sec. IV A. Actually this property holds true for time-dependent solutions as well and for all times, provided that x_1 and x_2 are initially uncorrelated. We stress that these strong properties are secured by the linearity of the reaction

scheme [14]. In the presence of nonlinear steps, U and Ψ will be different and one will have to resort to Eq. (17) to account fully for the probabilistic properties of the system.

Interestingly, a kinetic potential generating the deterministic evolution can still be constructed. For an asymmetric L matrix, the transformation of variables involved in its construction can no longer be a congruent transformation since it would then be noninvertible. Rather, one should resort to a similarity transformation diagonalizing the matrix of coefficients in Eq. (27) or (29) which is always legitimate as long as the eigenvalues of the matrix are distinct. On the other hand, under the transformation of variables necessary to cast the deterministic part of the evolution in a variational form, the diffusion part of the Fokker-Planck equation will keep an anisotropic structure similarly to Eq. (14a). This entails that the kinetic potential will no longer determine entirely the probabilistic properties of the system.

V. CONCLUSIONS

It has long been known that, in its most general setting, the evolution of a multivariate system subjected to nonequilibrium constraints cannot be cast in a variational form [4]. In the present work we have identified conditions under which this general trend can be overruled and illustrated them on two classes of phenomena: phase transitions in the presence of intermediate metastable phases and chemical reactions giving rise to two or more intermediate species. Much of the work dealt with nonlinear systems in which the thermodynamic fluxes and forces are linked by a symmetric positive definite Onsager matrix and the random forces satisfy a fluctuation-dissipation relation. For such systems a kinetic potential generating both the deterministic part of the evolution and the probability distribution in an appropriate set of variables has been derived and compared to the thermodynamic potential descriptive of the equilibrium state. When detailed balance does not hold, a kinetic potential may still exist insofar as the deterministic (mean field) part of the evolution is concerned. Typically, however, the fluctuations will not be generated by this potential but, rather, by a stochastic potential satisfying an equation of the form of (17).

Casting the evolution in the form of Eq. (4) or (13) as derived in our work is of special interest when studying fluctuation-induced transitions between states, as it provides the basis of a straightforward generalization of Kramers' theory to multivariate systems [11,12]. We stress that the variational property expressed by these equations is different from the one explored by, among others, Graham and Tel [15] (see also Ref. [5], Chap. 6). In this latter work the drift term of the Fokker-Planck equation is taken to be the sum of a contribution similar to the first term of the right-hand side

of Eqs. (2), plus an extra term required to be orthogonal to the gradient of the potential U.

Our results clarify further the limits beyond which a description of near-equilibrium states in terms of thermodynamic potentials like the free energy needs to be replaced by one exhibiting different types of state function. Since dissipation is the most ubiquitous signature of nonequilibrium, a natural question is whether these latter functions—referred to throughout as kinetic potentials—are related to the entropy production, which measures the amount of dissipation released in the system. The answer is negative, as can be seen already in the first-order reaction scheme of Eq. (24). Utilizing Eq. (28) and switching to excess variables around the reference state, one obtains straightforwardly the following expression for the excess entropy production:

$$\frac{1}{2}\delta^2\sigma = \frac{k_{-1}+k_2}{X_{1s}}x_1^2 + \frac{k_3+k_{-2}}{X_{2s}}x_2^2 - \left(\frac{k_2}{X_{2s}}+\frac{k_{-2}}{X_{1s}}\right)x_1x_2.$$

(39)

Comparing with expression (30b) for the (excess) thermodynamic potential U and keeping in mind that the kinetic potential is the transform of U upon the congruent transformation of Eq. (36), we conclude that the excess entropy production in the new variables is different from the kinetic potential. Nonlinearities can only accentuate this difference, since the "kinetic" nonlinearities appearing in the rate equations are generally different from the "thermodynamic" ones, as manifested at the level of the entropy production.

In the light of the above established limitations of the state functions of traditional thermodynamics, a natural extension of this work would be to seek for a generalized thermodynamic formalism built on the kinetic and the stochastic potentials. It would also be of interest to reconsider the questions raised herein from the standpoint of fluctuation theorems and large-deviation results reported in the recent literature [16]. An interesting step in this direction, suggesting how traditional thermodynamics needs to be extended to account for some key properties of nonequilibrium states, is described in Ref. [17].

Finally, yet another direction in the search for a variational formulation finds its origin in the maximum entropy principle [18]. Some interesting aspects have been explored recently in the literature [19], but the whole approach is different in spirit from and beyond the scope of the present work.

[1] G. Nicolis and I. Prigogine, *Self-Organization in Nonequilibrium Systems* (Wiley, New York, 1977); H. Haken, *Synergetics: An Introduction* (Springer, Berlin, 1977).

[2] L. Landau and E. Lifshitz, *Statistical Physics* (Pergamon, Oxford, 1959).

[3] S. De Groot and P. Mazur, *Nonequilibrium Thermodynamics* (North-Holland, Amsterdam, 1962).

[4] I. Prigogine, *Introduction to Thermodynamics of Irreversible Processes* (Wiley, New York, 1961); P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability and Fluctuations* (Wiley, London, 1971).

[5] H. Risken, *The Fokker-Planck Equation* (Springer, Berlin, 1984).

[6] R. Kubo, K. Matsuo, and K. Kitahara, J. Stat. Phys. **9**, 51 (1973).

[7] P. ten Wolde and D. Frenkel, Science **277**, 1975 (1997).

[8] P. Vekilov, Cryst. Growth Des. **4**, 671 (2004).

[9] G. Nicolis and C. Nicolis, Physica A **323**, 139 (2003); **351**, 22 (2005).

[10] J. F. Lutsko and G. Nicolis, Phys. Rev. Lett. **96**, 046102 (2006); V. Talanquer and D. Oxtoby, J. Chem. Phys. **109**, 223 (1998).

[11] B. Matkowsky and Z. Schuss, SIAM J. Appl. Math. **33**, 365 (1977).

[12] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).

[13] G. Nicolis, *Introduction to Nonlinear Science* (Cambridge University Press, Cambridge, U.K., 1995).

[14] G. Nicolis and A. Babloyantz, J. Chem. Phys. **51**, 2632 (1969).

[15] R. Graham and T. Tel, Phys. Rev. Lett. **52**, 9 (1984).

[16] See, e.g., C. Bustamante, J. Liphardt and F. Ritort, Phys. Today **58** (7), 43 (2005).

[17] H.-J. Woo, Europhys. Lett. **64**, 627 (2003).

[18] E. T. Jaynes, Am. J. Phys. **33**, 391 (1965).

[19] Q. A. Wang, e-print arXiv:0712.2583.